Internship report - Coherent and squeezed states in quantum optics, and quantization of the electromagnetic field.

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1 Introduction

At the intersection of optics and quantum physics, quantum optics is an active area of research in physics. Indeed, by combining the strengths of both, it promises a wide range of applications including secure telecommunication and perhaps even quantum computers. This document aims to summarize the work done during the "Immersion in a research laboratory" internship supervised by Pr. Hans R. JAUSLIN throughout the semester. Subjects covered were mainly theoretical, but experimental techniques were also discussed. This document has three main focuses, discussed in the order they were studied. We begin with a theoretical section that introduce the concept of minimum uncertainty states. We continue with an experiment-related section that introduce the principle of photon detection and two experimental methods used in quantum optics experiments. We conclude with the classical Maxwell's equations and a discussion towards the quantization of the electromagnetic field.

2 Minimum uncertainty states

Minimum uncertainty states are a special type of quantum states that are used for the detection of gravitational waves for instance, since they allow managing uncertainty of an observable, and then allowing more accurate measurements on it. In this first section, we define such states and formulate some of their properties.

2.1 First formulation

As its name suggests, a minimum uncertainty state ψ with respect to two self-adjoint operators $\hat{A}$ and $\hat{B}$ reaches the bound of HEISENBERG's inequality :

$$\Delta_\psi \hat{A} \Delta_\psi \hat{B} = \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle_\psi \right| \tag{1}$$

where the expectation value $\langle \hat{A} \rangle_\psi$ and the variance $\Delta_\psi \hat{A}$ are defined as

$$\langle \hat{A} \rangle_\psi := \langle \psi | \hat{A} | \psi \rangle \tag{2}$$

$$(\Delta_\psi \hat{A})^2 := \langle \psi | \hat{A}^2 | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2 . \tag{3}$$

Among its properties, a minimum uncertainty state ψ has its variances linked by a parameter ρ, and is an eigenvector of $\hat{A} + i\rho\hat{B}$:

$$\Delta_\psi \hat{A} = |\rho| \Delta_\psi \hat{B}, \tag{4}$$

$$(\hat{A} + i\rho\hat{B}) |\psi\rangle = (a + i\rho b) |\psi\rangle . \tag{5}$$

These properties allow one to calculate the value of variances for instance, but it also permits to categorize the minimum uncertainty state, as we will see later. In order to define coherent states and squeezed states, we have to define a reference harmonic oscillator, in particular its Hamiltonian and the associated annihilation operator :

$$\hat{H}_\omega = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 \tag{6}$$

$$\hat{a} := \frac{1}{\sqrt{2\hbar}}\left((m\omega)^{1/2}\hat{q} + i(m\omega)^{1/2}\hat{p}\right). \tag{7}$$

Relative to this harmonic operator, a coherent state is an eigenvector of the annihilation operator,

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\ . \tag{8}$$

For a coherent state, $\rho = \frac{1}{m\omega}$, and position and momentum variances are given by

$$\Delta_{|\alpha\rangle}q = \sqrt{\frac{\hbar}{2m\omega}} \qquad\qquad \Delta_{|\alpha\rangle} = \sqrt{\frac{\hbar m\omega}{2}}\,. \tag{9}$$

One can prove that if a measurement of the number of photons in a coherent state $|\alpha\rangle$ is made at time t, the probability of obtaining n photons will follow the Poisson distribution:

$$P_n = |\,\langle\alpha|\,(|n\rangle\langle n|)\,|\alpha\rangle\,|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\,. \tag{10}$$

Squeezed states are, as coherent states, minimum uncertainty states and also defined with respect to a reference harmonic oscillator, but the parameter ρ is different :

$$\rho \neq \frac{1}{m\omega}\,. \tag{11}$$

Note that most of the minimum uncertainty states are squeezed with respect to a given reference system.

2.2 Abstract formulation

The preceding formulation is interesting to understand what is a coherent state for example, but the need to define a reference harmonic oscillator is not convenient. For this reason, we develop here an abstract formulation, that only uses creation-annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ of a single mode of a general boson field, where we postulate

$$[\hat{a},\hat{a}^\dagger] = \mathbb{1}\,. \tag{12}$$

The observables we will discuss below are the quadrature operators, defined as

$$\hat{X} := \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger) \qquad \hat{Y} := -i\frac{1}{\sqrt{2}}(\hat{a}-\hat{a}^\dagger) \tag{13}$$

Coherent states are still defined as eigenvectors of the annihilation operator, and squeezed states are now defined as eigenvectors of the following operator :

$$\hat{b}_\nu = \frac{1}{\sqrt{2}}\left(\sqrt{\nu}\hat{X} + i\frac{1}{\sqrt{\nu}}\hat{Y}\right) \quad \text{with} \quad \nu \in \mathbb{R}^+ \backslash \{1\}\,. \tag{14}$$

We define two operators, that have a broader scope than that presented in this report: the displacement operator $\hat{D}_\alpha$ and the squeezing operator $\hat{S}_\xi$:

$$\hat{D}_\alpha = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha\hat{a}} = e^{|\alpha|^2/2} e^{-\alpha^*\hat{a}} e^{\alpha\hat{a}^\dagger} \tag{15}$$

$$\hat{S}_\xi = e^{\frac{1}{2}\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2}\right)} \quad \text{with} \quad \xi = re^{i\theta}\,. \tag{16}$$

As its name suggests, the displacement operator translates states, but a more interesting property is that it creates coherent states when applied to the vacuum state. The following association of the displacement operator and the squeezing operator can create squeezed states:

$$|\xi,\alpha\rangle = \hat{S}_\xi \hat{D}_\alpha |\emptyset\rangle\,. \tag{17}$$

States constructed with the squeezing operator (17) are not necessary minimum uncertainty states. That doesn't mean they aren't important: for high-precision measurements such as gravitational wave detection, only the uncertainty in the observable being measured is important.

3 Experimental techniques in quantum optics

In this section, we discuss experiment-related material. First, we explain one technique to detect single photons, and in a second part we look into two experimental techniques that can be used in quantum optics, homodyne and heterodyne detection. For the historical anecdote, heterodyne detection was developed by Pierre CONNES, originally from Dijon.

3.1 Photon detection

The first concept we will discuss is maybe the most important and fundamental: photon detection. Indeed, a single photon can only interact with microscopic particles, and we want to be able to get the information related to that interaction at the macroscopic level. In others words, we need to "convert" microscopic quantum information into a macroscopic, classical information. This raises one of the most fundamental concepts, the one associated with the measurement postulate. This will not be discussed here, and we will focus on measurements methods.

There are many ways to detect a photon at the atomic scale, and we will explain here only one. A somewhat easy method is photoelectric detection in solid-state devices, composed of semiconductors. During its flight inside the sensor, the photon has a probability - increasing with the length of the sensor - to be absorbed and to excite an electron from the valance band to the conduction band. This phenomenon creates an electron-hole pair that is free to move over the material. Note that the hole doesn't need to follow the electron and can go in others directions.

Now that we have that electron-hole pair, we need to be able to amplify the signal up to the macroscopic level. To do so, an electric field is applied inside the semiconductor, that accelerates the electron in a direction - and the hole in the other. Acquiring momentum, the electron will hit others electrons in the valance band, and with enough energy, will promote that other electron to the conduction band. The process repeats a great many times, and at the end of the semiconductor, a macroscopic current can be detected with standard electronics. This is called avalanche breakdown.

Modern sensors of this kind can achieve very good performances: quantum efficiency can be up to 90%, and the response time around the nanosecond.

3.2 Heterodyne and homodyne detection

We now discuss two special methods of detection of light signal: heterodyne and homodyne detection. Since the light frequency is much too high to be measured and analyzed with present electronics, the idea is to combine the signal to be measured with a known and very stable source of light, called local oscillator.

To do so in heterodyne detection, we use a very unbalanced beam-splitter - very unbalanced since the signal is low, and we want as small loss as possible. A phenomenon analogous to "beats" in acoustics then appears and called heterodyne signal. Indeed, we introduce for the example two polarized monochromatic waves, the signal $\boldsymbol{E_s}$ and the local oscillator $\boldsymbol{E_L}$

$$\boldsymbol{E}_L(\boldsymbol{r},t) = \mathcal{E}_L e^{i(k_L x - \omega_L t)} \boldsymbol{\epsilon} \tag{18}$$
$$\boldsymbol{E}_s(\boldsymbol{r},t) = \mathcal{E}_s e^{i(k_s y - \omega_s t)} \boldsymbol{\epsilon} \tag{19}$$

where $\boldsymbol{e}$ is the polarization vector. Assuming $\omega_L \neq \omega_s$, the field coming, after the beam-splitter, on the sensor is

$$\boldsymbol{E} = \mathcal{E}_L \underline{r}\ e^{i(k_L y - \omega_L t)} \boldsymbol{\epsilon} + \mathcal{E}_s \underline{t}\ e^{i(k_s y - \omega_s t)} \boldsymbol{\epsilon} \tag{20}$$

where r and t are respectively reflection and transmission coefficients of the beam-splitter. According to the preceding discussion, $|t| \simeq 1$ and $|r| \ll 1$. But detectors are only sensitive to intensity. Assuming that the detector is situated on $y = 0$, one can show that the received intensity is

$$|\boldsymbol{E}|^2 \simeq (\mathcal{E}_s \underline{t})^2 + (\mathcal{E}_L \underline{r})^2 + 2\,\mathrm{Re}\left(\mathcal{E}_s \underline{t} \mathcal{E}_L^* \underline{r}^* e^{-i(\omega_s - \omega_L)t}\right)$$

where $\omega_{IF} \equiv \omega_s - \omega_L$ is called intermediate frequency. The two firsts terms can be canceled by using high-pass electronics filters. The last term is called heterodyne signal. By selecting appropriate local oscillator frequency, we can expect an heterodyne signal frequency in the radio range, i.e. $10^6 \sim 10^9$ Hz, that can be processed by standard electronics.

The main advantages of heterodyne frequency are that the resulting signal is linear in the local oscillator field, linear in the signal field, and the relative phase between local oscillator and the signal is preserved. Some applications are the measurement of non-classical states of light like squeezed states, increased resolution in microscopy techniques such as structured illumination microscopy, increased density of information in optical fibers and improved accuracy of atomic clocks.

Homodyne detection uses the same principle as heterodyne detection, with a beam-splitter and a local oscillator, but the latter has the same frequency as the source. One approach to do so is to split in two beams the local oscillator: one will be used as the local oscillator and the other as the source, passing through the element to be probed. Then, when the two signals recombine, they interfere and the result of this interference is measured on both arms of the beam splitter. Signals from these two sensors are then subtracted with a differential amplifier. One of the advantages of homodyne detection is that it's not sensitive to fluctuations of the laser used to create the initial beam, and is sensitive to both intensity and phase shift of the signal. Note that the beam splitter is usually balanced, i.e. transmission and reflection coefficients are equal. Homodyne detection can be used as a detection scheme for a quantum key distribution, a fundamental part of encrypted secure communications techniques.

4 Maxwell's equations

The most fundamental equations in physics describing the behavior of light, and more broadly electromagnetic fields, are Maxwell's equations. In this section, we present these equations from a classical perspective, then we explain the work to be done to quantize the electromagnetic field.

4.1 From the classical perspective

To lay the groundwork, we start with the definition of Maxwell's equations and the Coulomb gauge, in the framework of classical physics, from which these two concepts were initially developed.

Maxwell's equations link the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ with their sources: the electric charge density ρ and the current density $\boldsymbol{j}$, as summarized bellow in the international SI system:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \quad (21) \qquad \nabla \wedge \boldsymbol{B} = \mu_0 \boldsymbol{j} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad (23)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (22) \qquad \nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,. \quad (24)$$

As we will see later, it is convenient to rewrite these equations with two others fields: the vector potential $\boldsymbol{A}$ and the electric potential U, defined as follows

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A} \quad (25) \qquad -\nabla U = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{E} \,. \quad (26)$$

One can remark that $\boldsymbol{A}$ and U are not unique, and this observation leads to the concept of gauge transformation, the passage from a pair $(\boldsymbol{A}, U)$ to another $(\boldsymbol{A}', U')$, such that

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla f \quad (27) \qquad U' = U - \frac{\partial f}{\partial t} \,. \quad (28)$$

The Coulomb gauge, used in quantum optics and in the rest of this document, satisfies

$$\Delta U_C = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_c = 0 \quad (29)$$

and one can prove that a Coulomb gauge transformation can be done starting from any potentials $\boldsymbol{A}$ and U. Using the Coulomb gauge in the case of free space, i.e. without charges neither currents, Maxwell's equations reduce to

$$\nabla \cdot \boldsymbol{A} = 0 \quad (30) \qquad \nabla \wedge \boldsymbol{A} = \boldsymbol{B} \quad (32)$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} \quad (31) \qquad c^2 \Delta \boldsymbol{A} = \frac{\partial \boldsymbol{E}}{\partial t} \,. \quad (33)$$

These four classical equations will be used to develop the quantum formalism of the electromagnetic field, as we will see in the next section.

4.2 Towards the quantum theory of the electromagnetic field

My exploration of the quantization of the electromagnetic field in this internship almost ends here, but there is still some work to be done to achieve quantization. The concept of modes was also studied, but in this section, the main ideas towards the quantum theory of the electromagnetic field are discussed.

The concept of mode is used to project the classical electric and magnetic fields onto a finite number of them. The model is then rewritten in Hamiltonian form and with the Coulomb gauge. This condition implies redundancy in the canonical variables $\mathbf{\Pi}$ and $\boldsymbol{A}$, so a new set of coordinates is introduced : (p_κ, q_κ). With these variables, Maxwell's equations take the form of N harmonic oscillators, and can be thus quantized. A problem arises when one wants to define the model with an infinite number of degrees of freedom: mathematical problems arise, and a new space has to be introduced: the FOCK space. In this space, the creation and annihilation operators can be properly defined, and then the physical observables. It's only after all this work that the time evolution can be calculated, using the SCHRÖDINGER or HEISENBERG representation.

5 Conclusion

This work introduced fundamental concepts in quantum optics, some from a conceptual perspective, others in a mathematical way. This document does not include any proofs, but most of the properties studied where proven during the internship. A detailed description of the calculations and proofs is presented in an appendix. An application linking these different concepts discussed here is quantum communication: detection methods like homodyne detection are used and needs to be deeply studied in order to achieve the highest detection rates as possible, and squeezed states are increasingly used in this field of research due to their interesting properties.

This internship provided me the opportunity to develop my knowledge and my understanding in quantum optics, a subject I'm particularly interested in. I wish I would have more time to study the subject in depth, but this internship was already a very good introduction. I would like to thank Édouard HERTZ for taking care of the administrative formalities and making this internship possible. Last but not least, I warmly thank Hans

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